Antipodal Graphs of Diameter Three

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ABSTRACT

Distance-regular graphs of diameter three are of three (almost distinct) kinds: primitive, bipartite, and antipodal. An antipodal graph of diameter three is just an r-fold covering of a complete graph K_{k+1} for some $r \leqslant k$. Its intersection array and spectrum are determined by the parameters r, k together with the number c of 2-arcs joining any two vertices at distance two. Most such graphs have girth three. In this note we consider antipodal distance-regular graphs of diameter three and girth $\geqslant 4$. If $r = 2$, then the only graphs are "$K_{k+1,k+1}$ minus a 1-factor." We therefore assume $r \geqslant 3$. The graphs with $c = 1$ necessarily have $r = k$ and were classified in [3]. We prove the inequality $r - 2 > c^{1/2}$ (Theorem 2), list the feasible parameter sets when $c = 2$ or 3 (Corollary 1), and conclude that every 3-fold or 4-fold antipodal covering of a complete graph has girth three (Corollary 2).

There are three (almost distinct) kinds of distance-regular graphs with diameter three: *primitive*, *bipartite*, and *antipodal* (see, for example, [1]). These classes are disjoint but for the fact that the graphs "$K_{k+1,k+1}$ minus a 1-factor" are both bipartite and antipodal.

The antipodal graphs have an apparently simple structure—being especially regular "coverings" of complete graphs [1,3]. Thus, for example, the vertices of the icosahedron come in six diametrically opposite pairs, with each vertex in each pair adjacent to precisely one vertex in each of the other five pairs: the icosahedron is thus a 2-fold antipodal covering of K_6. The intersection array, and hence the spectrum, of such a graph is determined by three parameters: the degree k, the index r ($\geqslant 2$) of the covering, and the number c of 2-arcs joining any two vertices at distance two. An antipodal distance-regular graph with parameters k, r, c has intersection array $\{k,(r-1)c,1;\ 1,c,k\}$,

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and the minimum polynomial of its adjacency matrix A is just the characteristic polynomial of the matrix

$$B=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & k-1-(r-1)c & c & 0 \\ 0 & (r-1)c & k-c-1 & k \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

namely, $(x-k)(x+1)[x^2+(rc+1-k)x-k]$. The eigenvalues $x = k, -1, x_1, x_2$ of B, and their multiplicities as eigenvalues of A $[m(k)=1, m(-1)=k, m(x_i)=(r-1)(k+1)k/(k+x_i^2)\ (i=1,2)]$, can all be calculated in terms of the parameters k, r, c. Such a graph is called an "r-fold antipodal covering of K_{k+1}" and is said to be "of type $(r.K_{k+1})_c$."

For all their apparent simplicity, we have no satisfactory general methods for deciding of a given triple k, r, c whether there exist any graphs of type $(r.K_{k+1})_c$ or, if such graphs exist, how many nonisomorphic ones there are. The so-called "feasibility" conditions (see, for example, [1, F1–F6]) are surprisingly good at excluding more complicated intersection arrays (see, for example, [2]). But for the relatively simple class of antipodal graphs of diameter three, there are masses of "feasible" parameter sets whose exact status is unresolved.

In this note we are concerned with antipodal graphs of diameter three and *girth* $\geqslant 4$. Such graphs satisfy $(r-1)c=k-1$, and so are of "sporadic type" [1]. These graphs arise naturally in the classification of 1-homogeneous graphs [4]. We prove the following results.

THEOREM 1. *Let G be a graph of type $(r.K_{k+1})_c$ having girth $\geqslant 4$. Then, with respect to the parameters $b=r-1$ and c, the intersection array of G becomes $\{bc+1, bc, 1;\ 1, c, bc+1\}$ and one of the following holds:*

(i) $r=2$ *and G is the unique graph of type $(2.K_{k+1})_{k-1}$ for some k (namely the graph "$K_{k+1,k+1}$ minus a 1-factor");*

(ii) $r\geqslant 3$, $c=1$, *and either* (a) $r=6$ *and G is the unique graph of type $(6.K_7)_1$; or* (b) $r=56$ *and G is a (possibly nonexistent) graph of type $(56.K_{57})_1$;*

(iii) $r\geqslant 3$, $c\geqslant 2$, $c^2+4(bc+1)=d^2$ *is a perfect square, and* $b(bc+2)(c+d)/2d\in\mathbb{Z}$.

THEOREM 2. *Let G be a graph of type $(r.K_{k+1})_c$ having girth $\geqslant 4$. Then either* (a) $r=2$ *and $G=$"$K_{k+1,k+1}$ minus a 1-factor," or* (b) $r-2>c^{1/2}$.

COROLLARY 1.

(i) *A graph of type* $(r.K_{k+1})_2$ *with* $r \geqslant 3$ *and having girth* $\geqslant 4$ *is feasible if and only if* $r = 2n^2$, $k+1 = 4n^2$ *for some* $n \geqslant 2$.

(ii) *A graph of type* $(r.K_{k+1})_3$ *with* $r \geqslant 3$ *and having girth* $\geqslant 4$ *is feasible if and only if either* $r = 14$, $k+1 = 41$, *or* $r = 351$, $k+1 = 1055$.

By Theorem 2 a graph of type $(r.K_{k+1})_c$ with $r \geqslant 3$, $c \geqslant 4$ and having girth $\geqslant 4$ is feasible only if $r \geqslant 5$. Hence we obtain

COROLLARY 2. *Every 3-fold or 4-fold antipodal covering of a complete graph has girth three.*

Proof of Theorem 1. Let G be a distance-regular graph of type $(r.K_{k+1})_c$ with vertex set V, and suppose that G has girth $\geqslant 4$. For each vertex u in V, let $G_i(u)$ denote the set of vertices at distance i from u. Thus $r-1 = |G_3(u)|$.

Suppose $r = 2$, and let $G_3(u) = \{u'\}$ for each vertex u in V. Then G is bipartite and is isomorphic to "$K_{k+1,k+1}$ minus the 1-factor $\{\{u,u'\} : u \in V\}$"—the unique graph of type $(2.K_{k+1})_{k-1}$. Thus we may assume that $r \geqslant 3$.

Counting the number of edges between $G_1(u)$ and $G_2(u)$ in two different ways, we get $k(k-1) = (r-1)kc$. Thus in terms of the parameter $b = r-1$ ($\geqslant 2$), we see that $k = bc+1$, that G has intersection array $\{bc+1, bc, 1; 1, c, bc+1\}$, and that its adjacency matrix has minimum polynomial $[x-(bc+1)][x+1][x^2+cx-(bc+1)]$ and eigenvalues $bc+1, -1, x_1, x_2$, where $x_1, x_2 = [-c \pm \sqrt{c^2+4(bc+1)}\,]/2$.

Suppose $c = 1$. Then $r = b+1 = k \geqslant 3$ and G is of type $(k.K_{k+1})_1$. But then, by [3], either $k = r = 6$ and G is the unique graph of type $(6.K_7)_1$, or $k = r = 56$ and G is a (possibly nonexistent) graph of type $(56.K_{57})_1$. Thus we may assume that $c \geqslant 2$.

It remains to interpret the condition that $m(x_i) = b(bc+2)(bc+1)/(bc+1+x_i^2) \in \mathbb{Z}$, which is clearly necessary for the existence of a graph of type $((b+1).K_{bc+2})_c$ having girth $\geqslant 4$. This integrality condition implies first that $x_i^2 = [-c \pm \sqrt{c^2+4(bc+1)}\,]^2/4$ must be rational, and hence that $c^2+4(bc+1) = d^2$ for some positive integer d. If we label x_1, x_2 so that $x_1 < 0 < x_2$, then the condition $m(x_i) \in \mathbb{Z}$ $(i = 1,2)$ reduces to $m(x_2) = b(bc+2)(c+d)/2d \in \mathbb{Z}$. [For later use, note that d is even if and only if c is even, and that $(d, d+c) = (d,c) = (d^2,c^2)^{1/2} = (4(bc+1), c^2)^{1/2} = (4, c^2)^{1/2} = 2$ or 1 according as d is even or odd.] ∎

Proof of Theorem 2. If $b = r-1 = 1$, the assertion follows from Theorem 1(i). We therefore assume that $b = r-1 \geqslant 2$, and use the Krein condition [1, F6] to show that $c^{1/2} < b-1$. If $[1, v_1(x), v_2(x), v_3(x)]^t$ is the right eigenvector of B with eigenvalue x, then $v_1(x_i) = x_i$, $v_2(x_i) = [x_i^2 - (bc+1)]/c = -x_i$, and $v_3(x_i) = -1$. The Krein condition $q_{222} \geqslant 0$ (with $\lambda_2 = x_1$) now yields

$$1 + \frac{x_1^3}{(bc+1)^2} - \frac{x_1^3}{(bc+1)^2 b^2} - \frac{1}{b^2} \geqslant 0,$$

$$\frac{b^2-1}{b^2} + \frac{b^2-1}{b^2}\,\frac{x_1^3}{(bc+1)^2} \geqslant 0.$$

Hence $(bc+1)^2 \geqslant -x_1^3$. Substituting $x_1 = (-c-d)/2$, $8(bc+1)^2 = (d^2 - c^2)^2/2$, we get

$$(d-c)^2 \geqslant 2(c+d).$$

Thus $[d-(c+1)]^2 > 4c+1$, so $d > (4c+1)^{1/2} + c + 1$. Squaring, rearranging, squaring again, and canceling then yields $b(bc+1) > c^2 + 3 + 3bc$, whence $c^2 < (b-3)(bc+1) < (b^2 - 2b)c < (b-1)^2 c$. Hence $c^{1/2} < b-1$. ∎

Proof of Corollary 1. (i): Let $c = 2$, $b \geqslant 2$, and suppose that a graph G of type $((b+1).K_{2(b+1)})_2$ is feasible. Then $d^2 = 8(b+1)$ is a perfect square, so $b = 2m-1 \geqslant 2$ is odd. But then $d^2 = 4^2 m$, so $m = n^2$ for some $n \geqslant 2$. Conversely, if $c = 2$ and $b = 2n^2 - 1$ for any $n \geqslant 2$, then $k+1 = bc+2 = 4n^2$, $m(x_2) = (2n^2-1)n(2n+1) \in \mathbb{Z}$, and the Krein conditions are easily seen to be satisfied (since $b \gg c$). Hence a graph of type $(2n^2.K_{4n^2})_2$ is feasible for every $n \geqslant 2$. [When $n = 1$ we get the graph $(2.K_4)_2 \cong Q_3$—the cube, with $r = 2$, $b = 1$.]

(ii): Let $c = 3$, $b \geqslant 2$, and suppose that a graph G of type $((b+1).K_{3b+2})_3$ is feasible. Then $b - 1 > c^{1/2}$, so $b \geqslant 3$. Moreover $d^2 = 12b + 13$ is a perfect square, so $b = 2m-1 \geqslant 3$ is odd, and $d^2 = 24m + 1$ for some $m \geqslant 2$. Since d is odd, $(2d, d+c) = 2$. And $(d, b) = (d^2, b^2)^{1/2} = (12b+13, b^2)^{1/2} = 13$ or 1, according as $b = 13(13b'+1)$ for some integer b', or not. Similarly $(d, bc+2) = (d, 3b+2) = (d^2, (3b+2)^2)^{1/2} = (4(3b+2)+5, (3b+2)^2)^{1/2} = 5$ or 1, according as $3b+2 = 5(5b''+1)$ for some integer b'', or not. Since $d \geqslant 7$, the integrality condition $m(x_2) = b(3b+2)(d+3)/2d \in \mathbb{Z}$ gives rise to just two possibilities: $d = 13$, $b = 13$, or $d = 65$, $b = 351$, both of which are feasible. ∎

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